Disagreement in Signed Financial Networks

Monica Billio, Roberto Casarin, Michele Costola, and Lorenzo Frattarolo

Abstract We extend the study of rate of convergence to consensus of autonomous agents on an interaction network. In particular, we introduce antagonistic interactions and thus a signed network. This will allow to include the, previously discarded, sign information, in the analysis of disagreement on statistical financial networks.

1 Introduction

Given the threat to financial stability and the real economy, quantifying systemic risk is now investigated by scholars as well as policy makers. More recently, graph theoretic measures and in particular convergence of autonomous agents on the network to a consensus have been involved in the systemic risk measurement such as early warning indicator for banking crises [Billio et al., 2017]. In this paper, we propose a generalization of disagreement index introduced in [Billio et al., 2017]. Building on the lifting approach in [Hendrickx, 2014], we are able to apply results in

Monica Billio
Department of Economics, Ca' Foscari University of Venice, Dorsoduro 3246, 30123 Venice (Italy), e-mail: billio@unive.it

Roberto Casarin
Department of Economics, Ca' Foscari University of Venice, Dorsoduro 3246, 30123 Venice (Italy), e-mail: r.casarin@unive.it

Michele Costola
Research Center SAFE, House of Finance, Goethe University, Theodor-W.-Adorno-Platz 3 60323 Frankfurt am Main, Germany, e-mail: costola@safe.uni-frankfurt.de.
The author acknowledges financial support from the Marie Skłodowska-Curie Actions, European Union, Seventh Framework Program HORIZON 2020 under REA grant agreement n.707070.

Lorenzo Frattarolo
Department of Economics, Ca' Foscari University of Venice, Dorsoduro 3246, 30123 Venice (Italy), e-mail: lorenzo.frattarolo@unive.it

[Billio et al., 2017] to the lifted dynamics extending their scope to signed networks. This allows to consider a more general consensus dynamics and disagreement with antagonism.

2 Disagreement in unsigned directed network

The relationship between the eigenvalues of the directed Laplacian (Diplacian), introduced in [Li and Zhang, 2012], and the rate of convergence of autonomous agents on the network to a consensus was studied in [Billio et al., 2017]. The application of those techniques to financial networks could be understood as measuring persistence of disagreement that is "magnified when major events occur in financial markets" according to [Carlin et al., 2014]. Our approach considers a limited communication network [Parikh and Krasucki, 1990] among agents, approximated by statistical causality relationship between stocks returns as already done in [Billio et al., 2017]. Consider a graph with adjacency matrix A and elements a_{ij} and with out-degree diagonal matrix D with non zero elements $d_i^{out} = \sum_{j=1}^{n} a_{ij}$. We introduce the following discrete time autonomous agent system:

$$x_{it} = x_{it-1} + \frac{1}{2d_i^{out}} \sum_{j=1}^{n} a_{ij} \left(x_{jt-1} - x_{it-1} \right) \tag{1}$$

Similar systems, introduced by [DeGroot, 1974], are building blocks in models of belief evolution of bounded rational agents, with a persuasion bias [DeMarzo et al., 2003]. The model written has the vectorial form :

$$\mathbf{x}_t = \frac{1}{2} I_n + \frac{1}{2} \left(I_n - D^{-1} \left(D - A \right) \right) \mathbf{x}_{t-1} = P_L x_{t-1}$$

Where $\mathbf{x}_t = (x_{it}, \dots, x_{nt})$ is the state vector of the agents, $P = D^{-1}A$ is the transition probability matrix of the Markov chain associated with random walks on G, where the probability of transitioning from vertex i to vertex j, $p_{ij} = a_{ij}/d_i^{out}$ of a random walk starting at i and $P_L = (I_n - P)$ corresponds to the transition matrix of the lazy random walk introduced in [Chung, 2005].

If the graph is strongly connected, P_L is irreducible and aperiodic, according well known results, the system converge to a consensus with group decision value $\varphi' \mathbf{x}_0$. The group decision is conserved by the dynamics:

$$\varphi' \mathbf{x}_t = \varphi' P_L \mathbf{x}_{t-1} \varphi' \mathbf{x}_{t-1} = \alpha.$$

We define the disagreement vector and its law of motion

$$\begin{aligned} \xi_t &= \mathbf{x}_t - \alpha \mathbf{1} \\ \xi_t &= P_L \xi_{t-1} \end{aligned}$$

The disagreement dynamics allows us to study the convergence rate in this directed unsigned case to this decision value. We exploit the theoretical results on lazy random walks on strongly connected directed graphs due to [Chung, 2005] and [Li and Zhang, 2012]. In particular in [Li and Zhang, 2012] the Diplacian Γ and its decomposition of in symmetric and asymmetric part is introduced

$$\Gamma = \varphi^{1/2} (I - P) \varphi^{-1/2}, \, \Gamma = L + \Delta, \, L = \frac{\Gamma + \Gamma'}{2}, \, \Delta = \frac{\Gamma - \Gamma'}{2}.$$

According to theorem 3 in [Billio et al., 2017] speed of convergence is expressed in terms of λ_2 the second smallest eigenvalue of L and of the second largest singular value $\sigma_{n-1} (I_n - L)$ of $I_n - L$ and the largest singular value $\sigma_n (\Delta)$ of the skew-symmetric part of the diplacian Δ, as in the following equation:

$$\|\xi_t\| \leq \exp\left\{ \frac{1}{2} \left[\log\left(\frac{\max(\varphi)}{\min(\varphi)} \right) + \log(\mu)\, t \right] \right\} \|\xi_0\|$$
$$\mu = \frac{3}{4} - \frac{\lambda_2}{2} + \frac{\left(\sigma_{n-1}(I_n - L) + \sigma_n(\Delta)\right)^2}{4}.$$

3 The lifted dynamics and Disagreement in signed networks

In the previous section the a_{ij}'s were non negative. In view of an application to a network, based on a vector autoregression (VAR) or similar methodology, this could result in neglecting an important source of information coming from the coefficients sign. The signed framework that correspond to antagonistic interactions among agents can be transformed in the unsigned one by a clever lifting trick introduced in [Hendrickx, 2014]. Consider a signed network $a_{ij} \in \mathbb{R}$ and define $b_{ij} = \max(0, a_{ij})$, $c_{ij} = \max(0, -a_{ij})$, $d_i^{|out|} = \sum_{j=1}^{n} \left| a_{ij} \right|$. We study the following dynamics with antagonistic interactions

$$\begin{aligned} x_{it} &= x_{it-1} + \frac{1}{2d_i^{|out|}} \sum_{j=1}^{n} \left| a_{ij} \right| \left(\operatorname{sign}(a_{ij}) x_{jt-1} - x_{it-1} \right) \\ &= x_{it-1} + \frac{1}{2d_i^{|out|}} \sum_{j=1}^{n} b_{ij} \left(x_{jt-1} - x_{it-1} \right) + \frac{1}{2d_i^{|out|}} \sum_{j=1}^{n} c_{ij} \left(-x_{jt-1} - x_{it-1} \right) \end{aligned}$$

or

$$x_{it} - x_{it-1} = \frac{1}{2d_i^{|out|}} \sum_{j=1}^{n} b_{ij} \left(x_{jt-1} - x_{it-1} \right) + \frac{1}{2d_i^{|out|}} \sum_{j=1}^{n} c_{ij} \left(-x_{jt-1} - x_{it-1} \right)$$

If we call $y_{it} = -x_{it}$ we obtain an analogous law of motion for y_{it}

$$y_{it} - y_{it-1} = \frac{1}{2d_i^{|out|}} \sum_{j=1}^{n} c_{ij} \left(y_{jt-1} - y_{it-1} \right) + \frac{1}{2d_i^{|out|}} \sum_{j=1}^{n} b_{ij} \left(-y_{jt-1} - y_{it-1} \right)$$

The joint dynamics can be written

$$\begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \end{bmatrix} = \frac{1}{2} \begin{bmatrix} I_n & \mathbf{0} \\ \mathbf{0} & I_n \end{bmatrix} - \frac{1}{2} \begin{bmatrix} \left(D^{|out|}\right)^{-1} & \mathbf{0} \\ \mathbf{0} & \left(D^{|out|}\right)^{-1} \end{bmatrix} \begin{bmatrix} B & C \\ C & B \end{bmatrix} \begin{bmatrix} \mathbf{x}_{t-1} \\ \mathbf{y}_{t-1} \end{bmatrix} = P_L \begin{bmatrix} \mathbf{x}_{t-1} \\ \mathbf{y}_{t-1} \end{bmatrix}$$

Analogously to [Hendrickx, 2014] $\mathbf{x}_t'$ is a solution of (2) if and only if $\mathbf{z}_t$ is a solution of the classical discrete time consensus system (2). If the solution exist then applying the same methodology as the unsigned case, by defining the lifted transition probability

$$P = \begin{bmatrix} \left(D^{|out|}\right)^{-1} & \mathbf{0} \\ \mathbf{0} & \left(D^{|out|}\right)^{-1} \end{bmatrix} \begin{bmatrix} B & C \\ C & B \end{bmatrix}, \tag{2}$$

the corresponding decision vector and Laplacians we can use (2) to bound the speed of convergence of the lifted dynamics and thus a upper bound for consensus dynamics on a signed directed network.
As discussed for the unsigned case in [Billio et al., 2017],those results could be readily applied to build a disagreement index that includes the sign information and understand its role in the measurement of systemic risk.

References

[Billio et al., 2017] Billio, M., Casarin, R., Costola, M., and Frattarolo, L. (2017). Contagion dynamics on financial networks. *Mimeo*.

[Carlin et al., 2014] Carlin, B. I., Longstaff, F. A., and Matoba, K. (2014). Disagreement and asset prices. *Journal of Financial Economics*, 114(2):226 – 238.

[Chung, 2005] Chung, F. (2005). Laplacians and the cheeger inequality for directed graphs. *Annals of Combinatorics*, 9(1):1–19.

[DeGroot, 1974] DeGroot, M. H. (1974). Reaching a consensus. *Journal of the American Statistical Association*, 69(345):118–121.

[DeMarzo et al., 2003] DeMarzo, P. M., Vayanos, D., and Zwiebel, J. (2003). Persuasion bias, social influence, and unidimensional opinions. *The Quarterly Journal of Economics*, 118(3):909–968.

[Hendrickx, 2014] Hendrickx, J. M. (2014). A lifting approach to models of opinion dynamics with antagonisms. In *Decision and Control (CDC), 2014 IEEE 53rd Annual Conference on*, pages 2118–2123. IEEE.

[Li and Zhang, 2012] Li, Y. and Zhang, Z.-L. (2012). Digraph laplacian and the degree of asymmetry. *Internet Math.*, 8(4):381–401.

[Parikh and Krasucki, 1990] Parikh, R. and Krasucki, P. (1990). Communication, consensus, and knowledge. *Journal of Economic Theory*, 52(1):178–189.